

# MELKERSSON CONDITION ON SERRE SUBCATEGORIES

REZA SAZEEDEH AND RASUL RASULI

**ABSTRACT.** Let  $R$  be a commutative noetherian ring, let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of  $R$ ; and let  $\mathcal{S}$  be a Serre subcategory of  $R$ -modules. We give a necessary and sufficient condition by which  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  conditions. As an conclusion we show that over a artinian local ring, every Serre subcategory satisfies  $C_{\mathfrak{a}}$  condition. We also show that  $\mathcal{S}_{\mathfrak{a}}$  is closed under extension of modules. If  $\mathcal{S}$  is a torsion subcategory, we prove that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition. We prove that  $C_{\mathfrak{a}}$  condition can be transferred via rings homomorphism. As some applications, we give several results concerning with Serre subcategories in local cohomology theory.

## 1. INTRODUCTION

Throughout this paper,  $R$  is a commutative noetherian ring. We denote by  $R\text{-Mod}$  the category of  $R$ -modules of  $R$ -homomorphisms and also we denote by  $R\text{-mod}$  the full subcategory of finitely generated  $R$ -modules. All subcategories considered in this paper are full subcategories of  $R\text{-Mod}$ ; unless otherwise stated. A subcategory  $\mathcal{S}$  of  $R\text{-Mod}$  is called Serre if it is closed under taking submodules, quotients and extensions of modules.

Let  $\mathcal{S}$  be a Serre subcategory;  $\mathfrak{a}$  an ideal of  $R$ ;  $M$  an  $R$ -module and  $n \in \mathbb{N}$ . It is a natural question to ask when the local cohomology modules  $H_{\mathfrak{a}}^i(M)$  belongs to  $\mathcal{S}$  for all  $i < n$  (or for all  $i > n$ ). The known examples of  $\mathcal{S}$  in this area are  $R\text{-mod}$  and  $R\text{-art}$ , where  $R\text{-art}$  is the subcategory of artinian  $R$ -modules. The same questions can be arisen for graded local cohomology modules  $H_{R_+}^i(M)$ , where  $R$  is a graded ring,  $R_+$  is irrelevant ideal,  $M$  is a graded modules and  $i$  is a non-negative integer. In the case of graded local cohomology, dealing with these questions plays an important role in measuring the number of minimal generators of the components of graded local cohomolgy modules (cf. [BFT, BRS, S]).

The authors in [AM] gave an answer when  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition where  $\mathfrak{a}$  is an ideal of  $R$ . This condition had already been posed for the subcategory of artinian modules by L. Melkersson [M]. We notice that this condition can easily be satisfied on a Serre subcategory whenever it is closed under injective envelops, but T. Yoshizawa [Y] gave an example which shows that the converse is not valid in general.

Let  $\mathfrak{a}$  be an ideal of  $R$ . In this paper we are interested in the study of  $C_{\mathfrak{a}}$  condition for subcategories in some more general cases. Let  $\mathcal{S}$  be a subcategory and let  $\mathfrak{a}$  be an ideal of  $R$ . We show that if  $\mathcal{S}$  is closed under taking submodules and  $\mathcal{S}$  satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition, then it satisfies  $C_{\mathfrak{a}}$  condition. Moreover we show that, the converse holds if  $\mathcal{S}$  is Serre. Let  $\mathfrak{b}$  be another ideal of  $R$ . We show that if a subcategory  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  conditions, then it satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  condition. In case where  $\mathcal{S}$  is Serre, we find a necessary and sufficient condition by which  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  conditions. As a conclusion, we prove that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition whenever  $\mathcal{S}$  satisfies  $C_{\mathfrak{p}}$  condition for each minimal  $\mathfrak{p} \in \text{Min}(\mathfrak{a})$ . Furthermore, we show that over an artinian ring, every Serre subcategory satisfies  $C_{\mathfrak{a}}$  condition for every ideal  $\mathfrak{a}$  of  $R$ .

---

2000 *Mathematics Subject Classification.* 13C60, 13D45.

*Key words and phrases.* Serre subcategory, Melkersson condition, local cohomology.

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two Serre subcategories, let  $\mathfrak{a}$  be an ideal of  $R$ . If  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  and  $\mathcal{S}_1 \cap \mathcal{S}_2$  satisfy  $C_{\mathfrak{a}}$  condition, then we show that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  satisfy  $C_{\mathfrak{a}}$  condition too. We also show that  $\mathcal{S}_{\mathfrak{a}}$  is closed under taking extension of modules for every Serre subcategory  $\mathcal{S}$ . Furthermore, we prove that if  $\mathcal{S}$  is a subcategory which is closed under taking submodules and arbitrary direct sums, then  $\mathcal{S}_{\mathfrak{a}}$  is closed under arbitrary direct sums for each ideal  $\mathfrak{a}$  of  $R$ ; in particular, if  $\mathcal{S}$  is a torsion subcategory, then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition. Lastly, we show that  $C_{\mathfrak{a}}$  condition can be transferred via rings homomorphisms  $\varphi : R \rightarrow S$  where  $\mathfrak{a}$  is an ideal of  $R$  (cf. Theorems 2.18, 2.19).

## 2. THE MAIN RESULTS

We start this section by the following definitions.

**Definitions 2.1.** Let  $\mathcal{S}$  be a class of  $R$ -Mod, let  $M$  be an  $R$ -module and let  $\mathfrak{a}$  be an ideal of  $R$ . The class  $\mathcal{S}$  is said to satisfy  $C_{\mathfrak{a}}$  condition on  $M$  whenever  $\Gamma_{\mathfrak{a}}(M) = M$  and  $(0 :_{\mathfrak{a}} M) \in \mathcal{S}$  imply  $M \in \mathcal{S}$ .

Let  $\mathcal{D}$  be a class of  $R$ -modules. The class  $\mathcal{S}$  is said to satisfy  $C_{\mathfrak{a}}$  condition on  $\mathcal{D}$  whenever  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $M$  for every  $M \in \mathcal{D}$ .

We denote by  $\mathcal{S}_{\mathfrak{a}}$  the largest subclass of  $R$ -Mod such that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}$ . It is clear to see that  $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{a}}$ .

The class  $\mathcal{S}$  is said to satisfy  $C_{\mathfrak{a}}$  condition whenever  $\mathcal{S}_{\mathfrak{a}} = R$ -Mod and  $\mathcal{S}$  is said to be *closed under  $C_{\mathfrak{a}}$  condition* whenever  $\mathcal{S}_{\mathfrak{a}} = \mathcal{S}$ .

In order to illustrate the above definitions and more understanding, we give several examples of subcategories  $\mathcal{S}$ .

**Examples 2.2.** (i) Let  $R$  be domain and let  $\mathcal{S}_{tf}$  be the class of torsion-free modules. Then  $\mathcal{S}_{tf}$  satisfies  $C_{\mathfrak{a}}$  condition for each ideal  $\mathfrak{a}$  of  $R$ . Indeed, the case  $\mathfrak{a} = 0$  is clear. For each non-zero ideal  $\mathfrak{a}$  of  $R$ , if  $\Gamma_{\mathfrak{a}}(M) = M$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ , it is immediate to see that  $(0 :_M \mathfrak{a}) = \Gamma_{\mathfrak{a}}(M) = 0$ . Furthermore, let  $\mathcal{S}_{tors}$  be the class of torsion modules. Then it is evident to see that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for each ideal  $\mathfrak{a}$  of  $R$ .

(ii) Let  $\mathcal{S}$  be a Serre subcategory of  $R$ -mod. It follows from [Y, Proposition 4.3] that  $R$ -mod  $\subseteq \mathcal{S}_{\mathfrak{a}}$  for every ideal  $\mathfrak{a}$  of  $R$ .

(iii) Let  $(R, \mathfrak{m})$  be a local ring and let  $\mathcal{S} = R$ -mod. Then  $E(R/\mathfrak{m}) \in \mathcal{S}_{\mathfrak{m}}$  if and only if  $R$  is artinian. To be more precise, suppose that  $E(R/\mathfrak{m}) \in \mathcal{S}_{\mathfrak{m}}$ . Since  $\Gamma_{\mathfrak{m}}(E(R/\mathfrak{m})) = E(R/\mathfrak{m})$  and  $\text{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m} \in \mathcal{S}$ , the module  $E(R/\mathfrak{m})$  is finitely generated and so  $R$  is artinian. Conversely if  $R$  is artinian, then  $E(R/\mathfrak{m}) \in \mathcal{S} \subseteq \mathcal{S}_{\mathfrak{m}}$ .

We state the following proposition which gives some basic properties of  $C_{\mathfrak{a}}$  condition of classes of modules, where  $\mathfrak{a}$  is an ideal of  $R$ .

**Proposition 2.3.** Let  $\mathcal{S}, \mathcal{T}$  and  $\mathcal{U}$  be three classes of  $R$ -modules such that  $\mathcal{S} \subseteq \mathcal{T}$ , let  $\mathfrak{a}$  be an ideal of  $R$  and let  $\mathcal{S}$  satisfy  $C_{\mathfrak{a}}$  condition on  $\mathcal{T}$ . Then the following statements hold.

- (i) If  $\mathcal{T}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{U}$ , then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{U}$ .
- (ii) There is  $\mathcal{T}_{\mathfrak{a}} \subseteq \mathcal{S}_{\mathfrak{a}}$ . Moreover,  $\mathcal{S}_{\mathfrak{a}} = \bigcup_{\mathcal{T}} \mathcal{T}_{\mathfrak{a}}$ , where  $\mathcal{T}$  is taken over

$$\sum = \{\mathcal{T} \mid \mathcal{S} \subseteq \mathcal{T} \text{ and } \mathcal{S} \text{ satisfies } C_{\mathfrak{a}} \text{ condition on } \mathcal{T}\}.$$

- (iii)  $\mathcal{S}_{\mathfrak{a}}$  is closed under  $C_{\mathfrak{a}}$  condition.

*Proof.* (i) Let  $M \in \mathcal{U}$  be an  $R$ -module such that  $\Gamma_{\mathfrak{a}}(M) = M$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . Since  $\mathcal{S} \subseteq \mathcal{T}$  and  $\mathcal{T}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{U}$ , there is  $M \in \mathcal{T}$  and since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{T}$ , there is  $M \in \mathcal{S}$ . (ii) Let  $M \in \mathcal{T}_{\mathfrak{a}}$  and let  $M = \Gamma_{\mathfrak{a}}(M)$  such that  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . Then  $(0 :_M \mathfrak{a}) \in \mathcal{T}$  and since  $M \in \mathcal{T}_{\mathfrak{a}}$ , there is  $M \in \mathcal{T}$ . Now, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{T}$ , there is  $M \in \mathcal{S}$ ; and hence  $M \in \mathcal{S}_{\mathfrak{a}}$ . The second equality follows easily by the first claim. (iii) In view of the definition it is clear that  $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{a}} \subseteq (\mathcal{S}_{\mathfrak{a}})_{\mathfrak{a}}$  and  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}$  and also  $\mathcal{S}_{\mathfrak{a}}$  satisfies  $C_{\mathfrak{a}}$  condition on  $(\mathcal{S}_{\mathfrak{a}})_{\mathfrak{a}}$ . Therefore the part (i) implies that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $(\mathcal{S}_{\mathfrak{a}})_{\mathfrak{a}}$ ; and so the definition implies that  $\mathcal{S}_{\mathfrak{a}} = (\mathcal{S}_{\mathfrak{a}})_{\mathfrak{a}}$ .  $\square$

The following lemma can be useful in the proof of next results.

**Lemma 2.4.** *Let  $\mathfrak{a}$  be an ideal of  $R$ , let  $\mathcal{S}$  be a subcategory which is closed under taking submodules and let  $\mathcal{S}$  satisfy  $C_{\mathfrak{a}}$  condition. If  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ , then  $\Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$  for every ideal  $\mathfrak{b}$  with  $\mathfrak{a} \subseteq \mathfrak{b}$ .*

*Proof.* It is clear that  $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = \Gamma_{\mathfrak{b}}(M)$  and  $(0 :_{\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) = \Gamma_{\mathfrak{b}}((0 :_M \mathfrak{a}))$ . Since  $\mathcal{S}$  is closed under taking submodules, we have  $(0 :_{\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) \in \mathcal{S}$  and since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $\Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$ .  $\square$

We now show that for every ideal  $\mathfrak{a}$  of  $R$ , a Serre subcategory satisfies  $C_{\mathfrak{a}}$  condition if and only if it satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition.

**Proposition 2.5.** *Let  $\mathcal{S}$  be a subcategory which is closed under taking submodules. If  $\mathcal{S}$  satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition, then it satisfies  $C_{\mathfrak{a}}$  condition. Moreover, if  $\mathcal{S}$  is Serre, then the converse holds too.*

*Proof.* Let  $M$  be an  $R$ -module such that  $\Gamma_{\mathfrak{a}}(M) = M$ , and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . Then  $\Gamma_{\sqrt{\mathfrak{a}}}(M) = M$  and since  $(0 :_M \sqrt{\mathfrak{a}}) \subset (0 :_M \mathfrak{a})$ , the hypothesis implies that  $(0 :_M \sqrt{\mathfrak{a}}) \in \mathcal{S}$ . Now, this fact that  $\mathcal{S}$  satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition implies that  $M \in \mathcal{S}$ . For the converse, let  $\mathcal{S}$  be a Serre subcategory and for convenience we set  $\mathfrak{b} = \sqrt{\mathfrak{a}}$ . As  $R$  is noetherian, there exists a non-negative integer  $n$  such that  $\mathfrak{b}^n \subseteq \mathfrak{a}$ . Let  $M$  be an  $R$ -module such that  $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M) = M$  and  $(0 :_M \mathfrak{b}) \in \mathcal{S}$ . Consider the following exact sequence of modules

$$0 \rightarrow \mathfrak{b}/\mathfrak{b}^2 \rightarrow R/\mathfrak{b}^2 \rightarrow R/\mathfrak{b} \rightarrow 0 \quad (\dagger).$$

The module  $\mathfrak{b}/\mathfrak{b}^2$  is a finitely generated  $R/\mathfrak{b}$ -module and so for some  $m \in \mathbb{N}$  there exists the following exact sequence of  $R$ -modules

$$0 \rightarrow K \rightarrow (R/\mathfrak{b})^m \rightarrow \mathfrak{b}/\mathfrak{b}^2 \rightarrow 0.$$

Applying the functor  $\text{Hom}_R(-, M)$  to this exact sequence, we deduce that  $\text{Hom}_R(\mathfrak{b}/\mathfrak{b}^2, M) \in \mathcal{S}$ . Moreover, applying the functor  $\text{Hom}_R(-, M)$  to the exact sequence  $(\dagger)$  and using this fact that  $\mathcal{S}$  is Serre, we deduce that  $(0 :_M \mathfrak{b}^2) \cong \text{Hom}_R(R/\mathfrak{b}^2, M) \in \mathcal{S}$ . Repeating the similar manner many times, we get  $(0 :_M \mathfrak{b}^n) \in \mathcal{S}$ . Now, applying the functor  $\text{Hom}_R(-, M)$  to the exact sequence  $0 \rightarrow \mathfrak{a}/\mathfrak{b}^n \rightarrow R/\mathfrak{b}^n \rightarrow R/\mathfrak{a} \rightarrow 0$ , we get  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . Lastly, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $M \in \mathcal{S}$ .  $\square$

**Proposition 2.6.** *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of  $R$  and let  $\mathcal{S}$  be a subcategory satisfying  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  conditions. Then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  condition. In particular, if  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for every principal ideal  $\mathfrak{a}$ , then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for every ideal  $\mathfrak{a}$ .*

*Proof.* Let  $M$  be an  $R$ -module such that  $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = M$  and  $(0 :_M \mathfrak{a} + \mathfrak{b}) \in \mathcal{S}$ . It is clear that  $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M) = M$ . On the other hand, we have the following isomorphisms

$$(0 :_M \mathfrak{a} + \mathfrak{b}) \cong \text{Hom}(R/\mathfrak{a} + \mathfrak{b}, M) \cong \text{Hom}(R/\mathfrak{a}, \text{Hom}(R/\mathfrak{b}, M)) \cong (0 :_{(0 :_M \mathfrak{b})} \mathfrak{a})$$

which imply that  $(0 :_{(0 :_M \mathfrak{b})} \mathfrak{a}) \in \mathcal{S}$ . Furthermore, we have the following equalities

$$\Gamma_{\mathfrak{a}}((0 :_M \mathfrak{b})) = (0 :_{\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) = (0 :_M \mathfrak{b}).$$

Now, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition, we deduce that  $(0 :_M \mathfrak{b}) \in \mathcal{S}$ . On the other hand since  $\Gamma_{\mathfrak{b}}(M) = M$  and  $\mathcal{S}$  satisfies  $C_{\mathfrak{b}}$  condition, we deduce that  $M \in \mathcal{S}$ . The second assertion follows by an easy induction on the number of generators of  $\mathfrak{a}$ .  $\square$

The following easy lemma is useful in proof of the next theorem.

**Lemma 2.7.** *Let  $\mathfrak{a}$  be an ideal of  $R$ , let  $\mathcal{S}$  be a Serre subcategory; and let  $M \in \mathcal{S}$ . Then  $\text{Ext}_R^i(R/\mathfrak{a}, M) \in \mathcal{S}$  for each  $i \geq 0$ .*

*Proof.* Let  $\dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$  be a free resolution of  $R/\mathfrak{a}$  such that each  $F_i$  is finitely generated. As  $\mathcal{S}$  is Serre,  $\text{Hom}_R(F_i, M) \in \mathcal{S}$  for each  $i$ . Now, since  $\text{Ext}_R^i(R/\mathfrak{a}, M)$  is the quotient of submodules of  $\text{Hom}_R(F_i, M)$ , we deduce that  $\text{Ext}_R^i(R/\mathfrak{a}, M) \in \mathcal{S}$ .  $\square$

Now, we are ready to state one of the main results of this paper.

**Theorem 2.8.** *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of  $R$  and let  $\mathcal{S}$  be a Serre subcategory. Then the following statements are equivalent:*

- (i)  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  and  $C_{\mathfrak{a} \cap \mathfrak{b}}$  conditions;
- (ii)  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  and  $C_{\mathfrak{a}\mathfrak{b}}$  conditions;
- (iii)  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  conditions.

*Proof.* (i)  $\Leftrightarrow$  (ii). As  $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}$ , it follows from Proposition 2.5 that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a} \cap \mathfrak{b}}$  condition if and only  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}\mathfrak{b}}$  condition.

(ii)  $\Rightarrow$  (iii). We prove that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition and a correspondence proof holds for the ideal  $\mathfrak{b}$ . Let  $M$  be an  $R$ -module and  $M = \Gamma_{\mathfrak{a}}(M)$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . It is clear to see that  $(0 :_M \mathfrak{a} + \mathfrak{b}) \subseteq (0 :_M \mathfrak{a})$  and so  $(0 :_M \mathfrak{a} + \mathfrak{b}) \in \mathcal{S}$ . As  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  condition, it follows from Lemma 2.4 that  $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$ . Now, consider the following exact sequence of  $R$ -modules

$$0 \rightarrow \Gamma_{\mathfrak{b}}(M) \rightarrow M \rightarrow M/\Gamma_{\mathfrak{b}}(M) \rightarrow 0.$$

Since  $\mathcal{S}$  is Serre, it suffices to show that  $M/\Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$ . Applying the functor  $\text{Hom}_R(R/\mathfrak{a}, -)$  to the above exact sequence and using Lemma 2.7, we conclude that  $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) \in \mathcal{S}$ . We now prove that  $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) = (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}\mathfrak{b})$ . The inequality  $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) \subseteq (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}\mathfrak{b})$  is obvious. For, the other inequality, let  $m + \Gamma_{\mathfrak{b}}(M) \in (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}\mathfrak{b})$ . Then  $\mathfrak{a}\mathfrak{b}m \subseteq \Gamma_{\mathfrak{b}}(M)$  and so there exists  $n \in \mathbb{N}$  such that  $\mathfrak{b}^n(\mathfrak{a}\mathfrak{b}m) = 0$ . This implies that  $\mathfrak{a}m \subseteq \Gamma_{\mathfrak{b}}(M)$ ; and hence  $m + \Gamma_{\mathfrak{b}}(M) \in (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a})$ . Therefore  $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}\mathfrak{b}) \in \mathcal{S}$ . On the other hand the fact that  $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{b}}(M)) = M/\Gamma_{\mathfrak{b}}(M)$  implies that  $\Gamma_{\mathfrak{a}\mathfrak{b}}(M/\Gamma_{\mathfrak{b}}(M)) = M/\Gamma_{\mathfrak{b}}(M)$ . Now, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}\mathfrak{b}}$  condition, there is  $M/\Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$ .

(iii)  $\Rightarrow$  (ii). That  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  condition follows by Proposition 2.6. Let  $M$  be an  $R$ -module such that  $\Gamma_{\mathfrak{a}\mathfrak{b}}(M) = M$  and  $(0 :_M \mathfrak{a}\mathfrak{b}) \in \mathcal{S}$ . As  $(0 :_{\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq (0 :_M \mathfrak{a}) \subseteq (0 :_M \mathfrak{a}\mathfrak{b}) \in \mathcal{S}$  and  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ . Considering the following exact sequence of  $R$ -modules

$$0 \rightarrow \Gamma_{\mathfrak{a}}(M) \rightarrow M \rightarrow M/\Gamma_{\mathfrak{a}}(M) \rightarrow 0,$$

it suffices to show that  $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ . Applying the functor  $\text{Hom}_R(R/\mathfrak{b}, -)$  to the above exact sequence induces the following exact sequence of  $R$ -modules

$$\text{Hom}_R(R/\mathfrak{b}, M) \rightarrow \text{Hom}_R(R/\mathfrak{b}, M/\Gamma_{\mathfrak{a}}(M)) \rightarrow \text{Ext}_R^1(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)).$$

As  $(0 :_M \mathfrak{b}) \subseteq (0 :_M \mathfrak{ab}) \in \mathcal{S}$ , there is  $\text{Hom}_R(R/\mathfrak{b}, M) \cong (0 :_M \mathfrak{a}) \in \mathcal{S}$ ; moreover Lemma 2.7 implies that  $\text{Ext}_R^1(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ . Therefore, since  $\mathcal{S}$  is Serre, we have

$$(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{b}) \cong \text{Hom}_R(R/\mathfrak{b}, M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}.$$

On the other hand, we show that  $\Gamma_{\mathfrak{b}}(M/\Gamma_{\mathfrak{a}}(M)) = M/\Gamma_{\mathfrak{a}}(M)$ . Let  $m + \Gamma_{\mathfrak{a}}(M) \in M/\Gamma_{\mathfrak{a}}(M)$ . Since  $\Gamma_{\mathfrak{ab}}(M) = M$ , there exists a positive integer  $n$  such that  $(\mathfrak{ab})^n m = 0$ . Thus  $\mathfrak{b}^n m \subseteq \Gamma_{\mathfrak{a}}(M)$  and so  $\mathfrak{b}^n(m + \Gamma_{\mathfrak{a}}(M)) = 0$ . The last equality implies that  $m + \Gamma_{\mathfrak{a}}(M) \in \Gamma_{\mathfrak{b}}(M/\Gamma_{\mathfrak{a}}(M))$ . Lastly, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{b}}$  condition, we have  $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ .  $\square$

**Corollary 2.9.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and let  $\mathcal{S}$  be a Serre subcategory. If  $\mathcal{S}$  satisfies  $C_{\mathfrak{p}}$  condition for every minimal prime ideal  $\mathfrak{p}$  of  $\mathfrak{a}$ , then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition.*

*Proof.* In view of Proposition 2.5, it suffices to show that  $\mathcal{S}$  satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be minimal prime ideals of  $R$ . Then  $\sqrt{\mathfrak{a}} = \cap_{i=1}^n \mathfrak{p}_i$ . As  $\mathcal{S}$  satisfies  $C_{\mathfrak{p}_i}$  condition for each  $i$ , by applying an easy induction and using Theorem 2.8, we deduce that  $\mathcal{S}$  satisfies  $C_{\sqrt{\mathfrak{a}}}$  condition.  $\square$

**Corollary 2.10.** *Let  $\mathcal{S}$  be a Serre subcategory and  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  be maximal ideals. If  $\mathcal{S}$  satisfies  $C_{\prod_{i=1}^n \mathfrak{m}_i}$  condition, then it satisfies  $C_{\mathfrak{m}_i}$  condition for each  $i$ .*

*Proof.* It is straightforward to see that  $\mathcal{S}$  satisfies  $C_R$  condition. On the other hand  $\prod_{i=1, i \neq j}^n \mathfrak{m}_i + \mathfrak{m}_j = R$  for each  $j$ . Therefore, it follows from Theorem 2.8 that  $\mathcal{S}$  satisfies  $C_{\mathfrak{m}_j}$  condition for each  $j$ .  $\square$

The following corollary shows that over an artinian ring, every Serre subcategory satisfies  $C_{\mathfrak{a}}$  condition for every  $\mathfrak{a}$  ideals of  $R$ .

**Corollary 2.11.** *Let  $R$  be an artinian ring and let  $\mathcal{S}$  be a Serre subcategory. Then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for each ideal  $\mathfrak{a}$  of  $R$ .*

*Proof.* Let  $\text{Max}R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ . Then  $\sqrt{0} = \prod_{i=1}^n \mathfrak{m}_i$ . It is straightforward to show that  $\mathcal{S}$  satisfies  $C_0$  condition and so in view of Proposition 2.5, it satisfies  $C_{\prod_{i=1}^n \mathfrak{m}_i}$  condition. Now Corollary 2.10 implies that  $\mathcal{S}$  satisfies  $C_{\mathfrak{m}_i}$  condition for each  $i$ . Lastly, in view of Corollary 2.9 we conclude that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for each ideal  $\mathfrak{a}$  of  $R$ .  $\square$

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two subcategories of  $R\text{-Mod}$ . We denote by  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  a class of  $R\text{-Mod}$  consisting of all  $R$ -modules  $M$  such that there exists an exact sequence of  $R$ -modules  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  with  $M_i \in \mathcal{S}_i$  for  $i = 1, 2$ . We can also refer to  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  as the class of extension modules of  $\mathcal{S}_1$  by  $\mathcal{S}_2$ . A well-known example is the class of *minimax* modules  $\mathcal{M} = \langle R\text{-mod}, R\text{-art} \rangle$ , where  $R\text{-art}$  is the subcategory of artinian modules.

**Theorem 2.12.** *Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two Serre subcategories, let  $\mathfrak{a}$  be an ideal of  $R$ ; and let  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  and  $\mathcal{S}_1 \cap \mathcal{S}_2$  satisfy  $C_{\mathfrak{a}}$  condition. Then  $\mathcal{S}_1$  and  $\mathcal{S}_2$  satisfy  $C_{\mathfrak{a}}$  condition.*

*Proof.* We prove the claim for  $\mathcal{S}_1$  and the proof for  $\mathcal{S}_2$  is similar. Let  $M$  be an  $R$ -module such that  $M = \Gamma_{\mathfrak{a}}(M)$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}_1$ . As  $\mathcal{S}_1 \subseteq \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  and  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $M \in \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ . Then there is an exact sequence of  $R$ -modules  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  such that  $M_1 \in \mathcal{S}$  and  $M_2 \in \mathcal{S}_2$ . Since  $\mathcal{S}_1$  is Serre, it suffices to show that  $M_2 \in \mathcal{S}_1$ . Taking the functor  $\text{Hom}_R(R/\mathfrak{a}, -)$  of the above short exact sequence, we obtain the following exact sequence of  $R$ -modules

$$\text{Hom}_R(R/\mathfrak{a}, M) \rightarrow \text{Hom}_R(R/\mathfrak{a}, M_2) \rightarrow \text{Ext}_R^1(R/\mathfrak{a}, M_1).$$

It follows from Lemma 2.7 that  $\text{Ext}_R^1(R/\mathfrak{a}, M_1) \in \mathcal{S}_1$  and since  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are Serre, we have  $(0 :_{M_2} \mathfrak{a}) \cong \text{Hom}_R(R/\mathfrak{a}, M_2) \in \mathcal{S}_1 \cap \mathcal{S}_2$ . On the other hand, it is evident to see that  $\Gamma_{\mathfrak{a}}(M_2) = M_2$  and since  $\mathcal{S}_1 \cap \mathcal{S}_2$  satisfies  $C_{\mathfrak{a}}$  condition, there is  $M_2 \in \mathcal{S}_1$ .  $\square$

An immediate corollary can be given rise from the above theorem.

**Corollary 2.13.** *Let  $\mathcal{M}$  and  $\mathcal{F}$  be the classes of all minimax modules and all modules of finite length, respectively; and let  $\mathfrak{a}$  be an ideal of  $R$ . If  $\mathcal{M}$  and  $\mathcal{F}$  satisfy  $C_{\mathfrak{a}}$  condition, then  $R\text{-mod}$  satisfies  $C_{\mathfrak{a}}$  condition.*

*Proof.* If we consider  $\mathcal{S}_1 = R\text{-mod}$  and  $\mathcal{S}_2 = R\text{-art}$ , then it is evident to see that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are Serre;  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \mathcal{M}$  and  $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{F}$ . Now, the result follows immediately by the previous theorem.  $\square$

**Proposition 2.14.** *Let  $\mathcal{S}$ ,  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be three subcategories such that  $\mathcal{S}$  is Serre and let  $\mathfrak{a}$  be an ideal of  $R$ . If  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , then it satisfies  $C_{\mathfrak{a}}$  condition on  $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ .*

*Proof.* Let  $M \in \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$  be an  $R$ -module such that  $M = \Gamma_{\mathfrak{a}}(M)$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . Then there is an exact sequence  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  such that  $M_1 \in \mathcal{S}_1$  and  $M_2 \in \mathcal{S}_2$ . It is clear to see that  $\Gamma_{\mathfrak{a}}(M_i) = M_i$  for  $i = 1, 2$  and since  $\mathcal{S}$  is Serre we have  $(0 :_{M_1} \mathfrak{a}) \in \mathcal{S}$ . Now, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_1$ , we have  $M_1 \in \mathcal{S}$ . Applying the functor  $\text{Hom}_R(R/\mathfrak{a}, -)$  to the above exact sequence and using Lemma 2.7 we deduce that  $(0 :_{M_2} \mathfrak{a}) \cong \text{Hom}_R(R/\mathfrak{a}, M_2) \in \mathcal{S}$ . Since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_2$ , there is  $M_2 \in \mathcal{S}$  and finally since  $\mathcal{S}$  is Serre, we have  $M \in \mathcal{S}$ .  $\square$

**Corollary 2.15.** *Let  $\mathcal{S}$  be a Serre subcategory and let  $\mathfrak{a}$  be an ideal of  $R$ . If  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $R\text{-art}$ , then it satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{M}$ , where  $\mathcal{M}$  is the class of all minimax modules.*

*Proof.* It is straightforward to show that  $\mathcal{S} \cap R\text{-mod}$  is a Serre subcategory of  $R\text{-mod}$  and it follows from [Y, Proposition 4.3] that  $\mathcal{S} \cap R\text{-mod}$  satisfies  $C_{\mathfrak{a}}$  condition on  $R\text{-mod}$ . Now, one can easily check that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $R\text{-mod}$ . Now the result follows by Proposition 2.14 as  $\mathcal{M} = \langle R\text{-mod}, R\text{-art} \rangle$ .  $\square$

For each subcategory  $\mathcal{S}$  of  $R\text{-Mod}$ , we set  $\mathcal{S}^0 = \{0\}$  and  $\mathcal{S}^{n+1} = \langle \mathcal{S}^n, \mathcal{S} \rangle$ ; for  $n \in \mathbb{N}$ . Moreover, we set  $\langle \mathcal{S} \rangle_{\text{ext}} = \bigcup \mathcal{S}^n$ . It is clear to see that  $\langle \mathcal{S} \rangle_{\text{ext}}$  is closed under taking extension of modules.

The following theorem shows that if  $\mathcal{S}$  is a Serre subcategory of  $R\text{-Mod}$  and  $\mathfrak{a}$  is an ideal of  $R$ , then  $\mathcal{S}_{\mathfrak{a}}$  is closed under taking extension of modules.

**Theorem 2.16.** *Let  $\mathcal{S}$  be a Serre subcategory and let  $\mathfrak{a}$  be an ideal of  $R$ . Then  $\mathcal{S}_{\mathfrak{a}}$  is closed under taking extension of modules.*

*Proof.* As  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}$ , it follows from Proposition 2.14 that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}^2$ . Repeating this way we deduce that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}^n$  for each  $n \in \mathbb{N}$ . Therefore  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\langle \mathcal{S}_{\mathfrak{a}} \rangle_{\text{ext}}$ . On the other hand,  $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{a}} \subseteq \langle \mathcal{S}_{\mathfrak{a}} \rangle_{\text{ext}}$  and by the definition  $\mathcal{S}_{\mathfrak{a}}$  is the largest subcategory of  $R\text{-Mod}$  such that  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition on  $\mathcal{S}_{\mathfrak{a}}$ . Thus this fact implies that  $\mathcal{S}_{\mathfrak{a}} = \langle \mathcal{S}_{\mathfrak{a}} \rangle_{\text{ext}}$ .  $\square$

We recall from [St] that a Serre subcategory  $\mathcal{S}$  of  $R\text{-Mod}$  is torsion subcategory if it is closed under taking arbitrary direct sums of modules. As direct limit of a direct system of modules is a quotient of a direct sum of modules, a torsion subcategory is closed under taking direct limits. The following theorem shows that a torsion subcategory  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition for each ideal  $\mathfrak{a}$  of  $R$ .

**Theorem 2.17.** *Let  $\mathcal{S}$  be a subcategory which is closed under taking submodules and let  $\mathfrak{a}$  be an ideal of  $R$ . Then the following statements hold.*

- (i) *If  $\mathcal{S}$  is closed under taking arbitrary direct sums, then so is  $\mathcal{S}_{\mathfrak{a}}$ .*
- (ii) *If  $\mathcal{S}$  is a torsion subcategory, then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition.*

*Proof.* (i) Let  $\{M_i\}$  be a subclass of  $\mathcal{S}_{\mathfrak{a}}$ . Then we show that  $\coprod M_i \in \mathcal{S}_{\mathfrak{a}}$ . Let  $\coprod M_i = \Gamma_{\mathfrak{a}}(\coprod M_i)$  and  $(0 :_{\coprod M_i} \mathfrak{a}) \in \mathcal{S}$ . Since  $\mathcal{S}$  is closed under taking submodules, there is  $(0 :_{M_i} \mathfrak{a}) \in \mathcal{S}$ ; and moreover  $M_i = \Gamma_{\mathfrak{a}}(M_i)$  for each  $i$ . Thus  $M_i \in \mathcal{S}_{\mathfrak{a}}$  yields  $M_i \in \mathcal{S}$  for each  $i$ . Now, according to the hypothesis we have  $\coprod M_i \in \mathcal{S}$  and so by the definition of  $\mathcal{S}_{\mathfrak{a}}$  we have  $\coprod M_i \in \mathcal{S}_{\mathfrak{a}}$ .

(ii) Let  $M = \Gamma_{\mathfrak{a}}(M)$  and let  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . For every finitely generated submodule  $N$  of  $M$ , it is straightforward to see that  $\Gamma_{\mathfrak{a}}(N) = N$  and  $(0 :_N \mathfrak{a}) \in \mathcal{S} \cap R\text{-mod}$ . Now, since  $\mathcal{S} \cap R\text{-mod}$  satisfies  $C_{\mathfrak{a}}$  condition on  $R\text{-mod}$  by [Y, Proposition 4.3], we have  $N \in \mathcal{S}$ . Finally, since  $M$  is direct limit of its finitely generated submodules, the assumption implies that  $M \in \mathcal{S}$ .  $\square$

Let  $\phi : R \rightarrow S$  be a rings homomorphism. Each  $S$ -module  $M$  can be considered as an  $R$ -module and so we set an additive and faithful functor  $\phi_{\star} : S\text{-Mod} \rightarrow R\text{-Mod}$ . It is straightforward to see that if  $\phi_{\star}(\mathcal{S})$  is a Serre subcategory of  $R\text{-Mod}$ , then  $\mathcal{S}$  is a Serre subcategory of  $S\text{-Mod}$ . Moreover, the converse is valid if  $\phi$  is epic. The following theorem shows that if  $\mathfrak{a}$  is an ideal of  $R$ , then  $C_{\mathfrak{a}}$  condition can be transferred via rings homomorphism.

**Theorem 2.18.** *Let  $\phi : R \rightarrow S$  be a rings homomorphism, let  $\mathfrak{a}$  be an ideal of  $R$  and let  $\mathcal{S}$  be a subcategory of  $S\text{-Mod}$ . The subcategory  $\phi_{\star}(\mathcal{S})$  satisfies  $C_{\mathfrak{a}}$  condition if and only if  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}S}$  condition.*

*Proof.* Let  $M$  be an  $S$ -module such that  $M = \Gamma_{\mathfrak{a}S}(M)$  and  $(0 :_M \mathfrak{a}S) \in \mathcal{S}$ . It is clear that  $M = \Gamma_{\mathfrak{a}}(M)$  and  $(0 :_M \mathfrak{a}S) = (0 :_M \mathfrak{a}) \in \phi_{\star}(\mathcal{S})$ . Since  $\phi_{\star}(\mathcal{S})$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $M \in \phi_{\star}(\mathcal{S})$  and so  $M \in \mathcal{S}$ . The proof of converse is the similar.  $\square$

Let  $\phi : R \rightarrow S$  be a rings homomorphism. Then there is an additive functor  $- \otimes_R S : R\text{-Mod} \rightarrow S\text{-Mod}$ . For a subcategory  $\mathcal{S}$  of  $R\text{-Mod}$ , we define  $\mathcal{S} \otimes S = \{M \otimes_R S \mid M \in \mathcal{S}\}$  which is a class of  $S$ -Modules. We now have the following theorem.

**Theorem 2.19.** *If  $\phi : R \rightarrow S$  be a faithfully flat rings homomorphism, let  $\mathfrak{a}$  be an ideal of  $R$  and let  $\mathcal{S}$  be a subcategory of  $R\text{-mod}$ . If  $\mathcal{S} \otimes S$  satisfies  $C_{\mathfrak{a}S}$  condition, then  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition.*

*Proof.* Let  $M$  be an  $R$ -module such that  $M = \Gamma_{\mathfrak{a}}(M)$  and  $(0 :_M \mathfrak{a}) \in \mathcal{S}$ . It is evident to see that  $M \otimes_R S = \Gamma_{\mathfrak{a}S}(M \otimes_R S)$  and  $(0 :_{M \otimes_R S} \mathfrak{a}S) \in \mathcal{S} \otimes S$ . Now, since  $\mathcal{S} \otimes S$  satisfies  $C_{\mathfrak{a}S}$  condition,  $M \otimes_R S \in \mathcal{S} \otimes S$  and so there exists  $N \in \mathcal{S}$  such that  $M \otimes_R S = N \otimes_R S$ . As  $S$  is a flat  $R$ -module and  $N$  is a finitely generated  $R$ -module, there exists a canonical isomorphism of  $S$ -modules

$$\omega : \text{Hom}_R(N, M) \otimes_R S \rightarrow \text{Hom}_S(N \otimes_R S, M \otimes_R S).$$

Thus there exists an  $R$ -homomorphism  $u : N \rightarrow M$  such that  $\omega(u \otimes 1) = u \otimes 1_S = 1_{N \otimes_R S}$ . Now since  $S$  is a faithfully flat  $R$ -module,  $u$  is isomorphism and so  $M \in \mathcal{S}$ .  $\square$

### 3. APPLICATIONS TO LOCAL COHOMOLOGY

**Proposition 3.1.** *Let  $\mathfrak{a}$  be an ideal of  $R$ , let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{a}}$  condition and let  $M \in \mathcal{S}$  be a finitely generated  $R$ -module. Then  $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$  for each  $i$ .*

*Proof.* We proceed by induction on  $i$ . If  $i = 0$ , then the result is clear as  $M \in \mathcal{S}$ . Let  $i > 0$  and without loss of generality let  $\Gamma_{\mathfrak{a}}(M) = 0$ . The, there exists an element  $x \in \mathfrak{a} \setminus Z(M)$  and an exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ . Applying the functor  $H_{\mathfrak{a}}^i(-)$  yields the following exact sequence

$$H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M).$$

the induction hypothesis implies that  $H_{\mathfrak{a}}^{i-1}(M/xM) \in \mathcal{S}$  and so  $(0 :_{H_{\mathfrak{a}}^i(M)} x) \in \mathcal{S}$ . Therefor, since  $\mathcal{S}$  is Serre,  $(0 :_{H_{\mathfrak{a}}^i(M)} \mathfrak{a}) \in \mathcal{S}$ . Now, since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition, we have  $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ .  $\square$

**Corollary 3.2.** *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of  $R$ , let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{a}}$  and  $C_{\mathfrak{b}}$  condition and let  $M \in \mathcal{S}$  be a finitely generated  $R$ -module. Then all modules  $H_{\mathfrak{a}}^i(M), H_{\mathfrak{b}}^i(M), H_{\mathfrak{a}+\mathfrak{b}}^i(M), H_{\mathfrak{a}\mathfrak{b}}^i(M)$  lye in  $\mathcal{S}$  for all  $i$ .*

*Proof.* According to Theorem 2.8, the subcategory  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}+\mathfrak{b}}$  and  $C_{\mathfrak{a}\mathfrak{b}}$  conditions. Now, the result follows by Proposition 3.1.  $\square$

**Corollary 3.3.** *If  $\mathcal{S}$  is a torsion subcategory and  $M \in \mathcal{S}$ , then  $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$  for each  $i$ .*

*Proof.* Without loss of generality, we may assume that  $M$  is finitely generated and so the result follows by the previous proposition and Theorem 2.17.  $\square$

**Proposition 3.4.** *Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_t$  be maximal ideals and  $\mathfrak{a}$  be an arbitrary ideal of  $R$ , let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{a}}$  condition and let  $n$  be a non-negative integer such that  $\text{Supp}(H_{\mathfrak{a}}^i(M)) \subseteq \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$  for all  $i \leq n$  (note that  $n$  may be  $\infty$ ). Then  $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$  for all  $i \leq n$ .*

*Proof.* We proceed by induction on  $i$ . If  $i = 0$ , then  $\Gamma_{\mathfrak{a}}(M)$  is finite length and so  $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ . Let  $i > 0$  and suppose inductively that the result has been proved for all values smaller than  $i$  and all finitely generated  $R$ -modules and so we prove it for  $i$ . Now the result follows by a similar proof mentioned in Proposition 3.1.  $\square$

**Corollary 3.5.** *Let  $\mathfrak{m}$  be a maximal ideal, let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{m}}$  condition and let  $M$  be a finitely generated  $R$ -module. Then  $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$  for each  $i$ .*



*Proof.* The result follows by the previous proposition.  $\square$

**Proposition 3.6.** *Let  $\mathfrak{a}$  be an ideal of  $R$ , let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{a}}$  condition. Let  $M$  be a finitely generated  $R$ -module and  $n$  be a non-negative integer such that  $H_{\mathfrak{a}}^i(M)$  is minimax for all  $i < n$ . Then  $\Gamma_{\mathfrak{m}}(H_{\mathfrak{a}}^n(M)) \in \mathcal{S}$  for every maximal ideal  $\mathfrak{m}$  of  $R$ .*

*Proof.* According to [BN, Theorem 2.3], the  $R$ -module  $(0 :_{H_{\mathfrak{a}}^n(M)} \mathfrak{a})$  is finitely generated and so  $\Gamma_{\mathfrak{m}}((0 :_{H_{\mathfrak{a}}^n(M)} \mathfrak{a})) = (0 :_{\Gamma_{\mathfrak{m}}(H_{\mathfrak{a}}^n(M))} \mathfrak{a})$  is finite length. Thus  $(0 :_{\Gamma_{\mathfrak{m}}(H_{\mathfrak{a}}^n(M))} \mathfrak{a}) \in \mathcal{S}$  and since  $\mathcal{S}$  satisfies  $C_{\mathfrak{a}}$  condition,  $\Gamma_{\mathfrak{m}}(H_{\mathfrak{a}}^n(M)) \in \mathcal{S}$ .  $\square$

**Proposition 3.7.** *Let  $(R, \mathfrak{m})$  be a local ring, let  $\mathfrak{a}$  be an ideal of  $R$ , and let  $\mathcal{S}$  be a Serre subcategory satisfying  $C_{\mathfrak{a}}$  condition. If  $M$  is a finitely generated  $R$ -module of dimension  $n$ , then  $H_{\mathfrak{a}}^n(M) \in \mathcal{S}$ .*

*Proof.* We proceed by induction on  $n$ . if  $n = 0$ , then  $\Gamma_{\mathfrak{a}}(M)$  is of finite length and so there is nothing to prove in this case. Let  $n > 0$  and we may assume that  $\Gamma_{\mathfrak{a}}(M) = 0$ . Now the result follows by using a similar proof that mentioned in Proposition 3.1.  $\square$

For an  $R$ -module  $M$ , the *cohomological dimension* of  $M$  with respect to an ideal  $\mathfrak{a}$  is defined as  $\text{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : H_{\mathfrak{a}}^i(M) \neq 0\}$ .

The following result show that some quotients of top local cohomology modules may be belong to Serre subcategories.

**Proposition 3.8.** *Let  $(R, \mathfrak{m})$  be a local ring, let  $\mathfrak{a}$  be an ideal of  $R$ , let  $\mathcal{S}$  be a Serre subcategory, and let  $M$  be a finitely generated  $R$ -module with  $c(\mathfrak{a}, M) = n$ . Then  $H_{\mathfrak{a}}^n(M)/\mathfrak{m}H_{\mathfrak{a}}^n(M) \in \mathcal{S}$ .*

*Proof.* we proceed by induction on  $n$ . If  $n = 0$ , the module  $\Gamma_{\mathfrak{a}}(M)/\mathfrak{m}\Gamma_{\mathfrak{a}}(M)$  is of finite length and so there is nothing to prove in this case. Let  $n > 0$  and we may assume that  $\Gamma_{\mathfrak{a}}(M) = 0$  and so there exists an element  $x \in \mathfrak{a} \setminus Z(M)$  and an exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ . It follows from [DNT] that  $c(\mathfrak{a}, M/xM) \leq c(\mathfrak{a}, M)$  and so there is the following exact sequence

$$H_{\mathfrak{a}}^{n-1}(M/xM) \rightarrow H_{\mathfrak{a}}^n(M) \xrightarrow{x} H_{\mathfrak{a}}^n(M) \rightarrow 0 (*).$$

As each element of  $H_{\mathfrak{a}}^{n-1}(M)$  is  $x$ -torsion, the equality  $H_{\mathfrak{a}}^{n-1}(M/xM) = 0$  implies that  $H_{\mathfrak{a}}^n(M) = 0$  which is a contradiction. Therefore  $c(\mathfrak{a}, M/xM) = n - 1$ . Now, using the inductive hypothesis, we conclude that  $H_{\mathfrak{a}}^{n-1}(M/xM)/\mathfrak{m}H_{\mathfrak{a}}^{n-1}(M/xM) \in \mathcal{S}$ . Applying the functor  $R/\mathfrak{m} \otimes_R -$  to the exact sequence  $(*)$ , we have  $R/\mathfrak{m} \otimes_R (0 :_{H_{\mathfrak{a}}^n(M)} \mathfrak{a}) \in \mathcal{S}$ , moreover there is an epimorphism  $R/\mathfrak{m} \otimes_R (0 :_{H_{\mathfrak{a}}^n(M)} \mathfrak{a}) \twoheadrightarrow H_{\mathfrak{a}}^n(M)/\mathfrak{m}H_{\mathfrak{a}}^n(M)$  which completes the proof.  $\square$

## REFERENCES

- [AM] M. Aghapournahr and L. Melkersson, *Local Cohomology and Serre subcategories*, J. Algebra, **320**(2008), 1275-1287.
- [BN] K. Bahmanpour and R. Naghipour, *On the cofiniteness of local cohomology modules*, Proc. Amer. Math. Soc, **136**(2008), 2359-2363.
- [BFT] M. Brodmann, S. Fumasoli and R. Tajarod, *Local cohomology over homogeneous rings with one-dimensional local base ring*, Proc. Amer. Math. Soc, **131**(2003), 2977 - 2985.
- [BRS] M. Brodmann, F. Rohrer and R. Sazeedeh, *Multiplicities of graded components of local cohomology modules*, J. Pure Appl. Algebra, **197**(2005), 249-278.

- [DNT] K. Divaani-Aazar, R. Naghipour, AND M. Tousi, *Cohomological dimension of certain algebraic varieties*, Proc. Amer. Math. Soc, **130**(2002),3537-3544.
- [M] L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, Math. Proc. Cambridge Philos. Soc. **107**(1990), 267-271.
- [S] R. Sazeedeh, *Artinianess of graded local cohomology modules*, Proc. Amer. Math. Soc, **135**(2007), 2339-2345.
- [St] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathmatischen Wissenschaften, vol. **217**, Springer-Verlag 1975.
- [Y] T.Yoshizawa, *An example of Melkersson subcategory which is not closed under injective hulls*, arXiv:1011.1663v2 [math.AC] 9 Nov 2010.

DEPARTMENT OF MATHEMATICS, URMIA UNIVERSITY, P.O.Box: 165, URMIA, IRAN-AND, SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P. O. Box: 19395-5746, TEHRAN, IRAN

*E-mail address:* `rsazeedeh@ipm.ir`

MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, PAYAME NOOR UNIVERSITY(PNU), TEHRAN, IRAN

*E-mail address:* `rasulirasul@yahoo.com`